VCU, Department of Computer Science

CMSC 302

Functions

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On to section 2.3… Functions

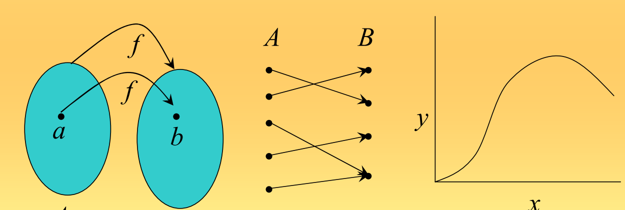
- From calculus, you are familiar with the concept of a real-valued function *f*, which assigns to each number $x \in \mathbb{R}$ a particular value $y=f(x)$, where $y \in \mathbb{R}$.
- But, the notion of a function can also be naturally generalized to the concept of **assigning elements of *any* set to elements of *any* set.** (Also known as a ***map***.)

Function: Formal Definition

- Def. For any sets *A*, *B*, we say that a *function f from (or "mapping") A to B* ($f{:}A\rightarrow B$) is a particular assignment of exactly **one element** $f(x) \in B$ to **each element** $x \in A$, or *b* is a **unique map** of *a.*
- In other words, **in a function, every a∈A is associated with exactly one b∈B!**
- Some further generalizations of this idea:
  - A *partial* (non-*total*) function *f* assigns *zero or one* elements of *B* to each element $x \in A$.
  - Functions of *n* arguments; relations (ch. 8).

Graphical Representations

- Functions can be represented graphically in several ways:



Like Venn diagrams

Bipartite Graph

Plot

Some Function Terminology

- Def. Let $f:A \to B$, and $f(a)=b$ (where $a \in A$ & $b \in B$). Then
  - A is the *domain* of f.
  - B is the *codomain* of f.
  - b is the *image* of a under f.
  - a is a *pre-image* of b under f.
    - In general, b may have more than 1 pre-image.
  - The *range* $R \subseteq B$ of f is $R=\{b \mid \exists a\ f(a)=b\}$.

We also say the *signature* of f is $A \to B$.

Range versus Codomain

- Remarks.
- The range of a function might *not* be its whole codomain.
- The codomain is the set that the function is *declared* to map all domain values into.
- The range is the *particular* set of values in the codomain that the function *actually* maps elements of the domain to.

Range vs. Codomain - Example

- Ex. Suppose I declare to you that: "f is a function mapping students in this class to the set of grades {A,B,C,D,E}."
- At this point, you know f's codomain is: {A,B,C,D,E}, and its range is unknown!.
- Suppose the grades turn out all As and Bs.
- Then the range of f is {A,B}, but its codomain is still {A,B,C,D,E}!.

Operators (general definition)

- Def. An *n-ary operator over* (or *on*) the set S is any function from the set of ordered n-tuples of elements of S, to S itself.
- Ex.
  $\neg$ can be seen as a unary operator, and $\wedge, \vee$ are binary operators on S.
- Ex. $\cup$ and $\cap$ are binary operators on the set of all sets.

Constructing Function Operators

- If • ("dot") is any operator **over *B***, then we can extend • to also denote an operator over <u>functions</u> $f:A \to B$.
- Ex. Given any binary operator $\bullet : B \times B \to B$, and functions $f,g:A \to B$, we define $(f \bullet g):A \to B$ to be the function defined by: $\forall a \in A, (f \bullet g)(a) = f(a) \bullet g(a)$.

Ex: $f(x) = x$, $g(x) = x + 3$; *then* $(f \bullet g)(x) = f(x) \bullet g(x) = x(x + 3)$

And same is valid for + , summation operator $(f + g)(x) = f(x) + g(x)$

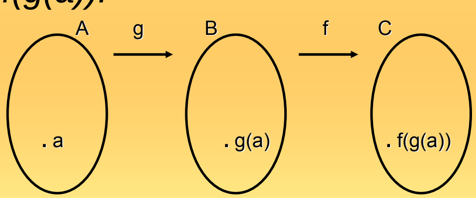
Function Operator Example

- $+,\times$ ("plus","times") are binary operators over $\mathbb{R}$. (Normal addition & multiplication.)
- Therefore, we can also add and multiply *functions*
- Def. Let f, g: $\mathbb{R} \to \mathbb{R}$.

  $(f + g)$: $\mathbb{R} \to \mathbb{R}$, where $(f + g)(x) = f(x) + g(x)$

  $(f \times g)$: $\mathbb{R} \to \mathbb{R}$, where $(f \times g)(x) = f(x) \times g(x)$

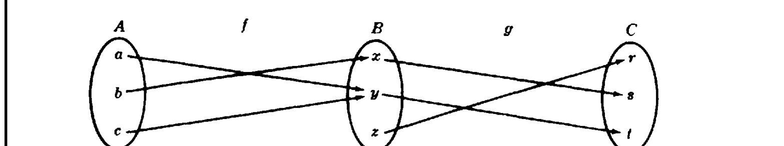
Function Composition Operator

- Def. Let $g:A \to B$ and $f:B \to C$. *The composition of f and g*, denoted by $f \circ g$, is defined by $(f \circ g) = f(g(a))$.
- 

- Remark. $\circ$ (like Cartesian $\times$, but unlike $+,\wedge,\cup$) is non-commuting. (Generally, $f \circ g \neq g \circ f$.)

- **Example** of a composition:

Let the functions $f: A \to B$ and $g: B \to C$ be defined by Fig. 3-9. $\boxed{\text{Find the composition function } g \circ f: A \to C.}$



- **Solution**:

$$(g \circ f)(a) = g(f(a)) = g(y) = t$$
$$(g \circ f)(b) = g(f(b)) = g(x) = s$$
$$(g \circ f)(c) = g(f(c)) = g(y) = t$$

Images of Sets under Functions

- Def. Let $f:A\rightarrow B$, and $S\subseteq A$.
  The *image* of S under f is simply the set of all images (under f) of the elements of S.
  $f(S) :\equiv \{f(s) \mid s\in S\}$
  $:\equiv \{b \mid \exists\ s\in S: f(s)=b\}$.
- Ex: Let A={a, b, c, d, e} and B={1, 2, 3, 4} with *f*(a) = 2, *f*(b) = 1, *f*(c) = 4, *f*(d) = 1 and *f*(e) = 1. The image of the subset S = {b, c, d} is the set *f*(S) = {1, 4}.

I stopped here last time

One-to-One Functions

- Def. A function is *one-to-one* (*1-1*), or *injective*, or *an injection*, iff every element of its **range** has *only* 1 pre-image.
  - Formally: given $f:A\rightarrow B$,
    "f is injective" $:\equiv (\neg\exists x,y: x\neq y \wedge f(x)=f(y))$.
- Only one element of the domain is mapped to any given one element of the range.
  - Domain & range have same cardinality. What about codomain?

May Be Larger

- Memory jogger: Each element of the domain is injected into a different element of the range.
  - Compare "each dose of vaccine is injected into a different patient."

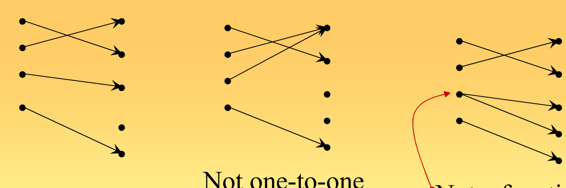
One-to-One Functions

- Each element of the domain is injected into a different element of the range.
- Ex: Determine whether the function *f* from {a, b, c, d} to {1, 2, 3, 4, 5} with *f*(a)=4, *f*(b)=5, *f*(c)=1 and *f*(d)=3 is one-to-one.

Yes

One-to-One Illustration

- Bipartite (2-part) graph representations of functions that are (or not) one-to-one:



One-to-one

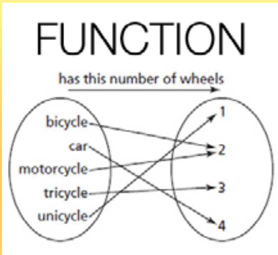
Not one-to-one

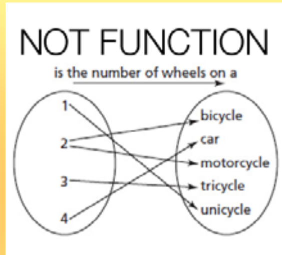
Not a function!

See slide 21 commenting NOT being a FUNCTION

Function definition

Ex 1:





Ex 2: Determine if the relation is a function.
{(1,2), (2,3), (3,4), (1,5)}

Answer: **No** it is a not a function because there is a **repetition of the domain**.
{(**1**,2), (2,3), (3,4), (**1**,5)}

Sufficient Conditions for 1-1ness

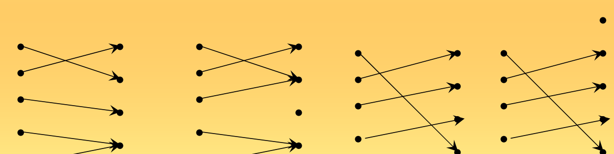
- For functions *f* **over numbers**, we say:
  - *f* is *strictly* (or *monotonically*) *increasing* iff $x>y \rightarrow f(x)>f(y)$ for all *x,y* in domain;
  - *f* is *strictly* (or *monotonically*) *decreasing* iff $x>y \rightarrow f(x)<f(y)$ for all *x,y* in domain;
- If *f* is either strictly increasing or strictly decreasing, then *f* is one-to-one.
- Ex.
- $f(x) = x^3$
- $f(x) = 1/x$

Onto (Surjective) Functions

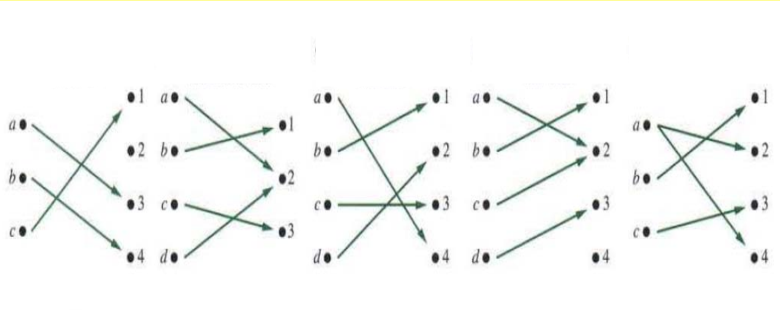
- Def. A function $f{:}A\rightarrow B$ is ***onto*** or ***surjective*** or *a surjection* iff its **range is equal to its codomain** ($\forall b\in B, \exists a\in A$: $f(a)=b$).
- Remark. An *onto* function maps the set *A* onto (over, covering) the *entirety* of the set *B*, not just over a piece of it.
- Ex. Let $f{:}\mathbb{R} \rightarrow \mathbb{R}$.
  - $f(x) = x^3$ is onto,
  - $f(x) = x^2$ is not onto. (Why?)

Illustration of Onto

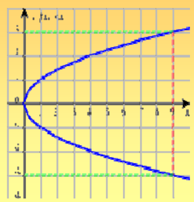
- Some functions that are, or are not, *onto* their codomains:



Onto (but not 1-1) | Not Onto (or 1-1) | Both 1-1 and onto | 1-1 but not onto



Comment on NOT being a FUNCTION



$y = f(x) = \sqrt{x}$

is not a function in the **proper** sense, but a **multi-valued function**!

Bijections

- Def. A function f is said to be *a* ***bijection***, (or *a* ***one-to-one correspondence***, or *reversible*, or *invertible*,) iff
  it is <u>both</u> **one-to-one** <u>and</u> **onto**.

- Def. For bijections $f{:}A\rightarrow B$, there exists an *inverse of* f, written $f^{-1}{:}B\rightarrow A$, which is the unique function such that $f^{-1} \circ f = I_A$
  - (where I_A is the identity function on A)

Inverse Function

- Definition: Let f be a <u>one-to-one correspondence</u> from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that f(a)=b. The inverse function of f is denoted by f^{-1}.

  Hence, $\boldsymbol{f^{-1}(b) = a}$ **when** $\boldsymbol{f(a) = b}$.

The Identity Function

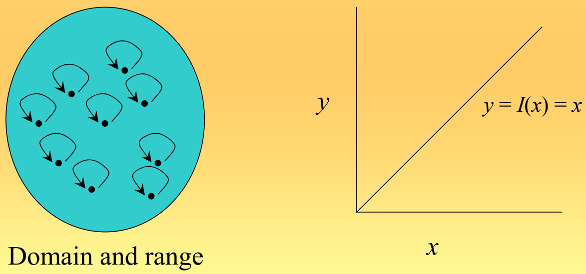
- Def. For any domain A, the *identity function* $I{:}A\rightarrow A$ (variously written, I_A, **1**, $\mathbf{1}_A$) is the unique function such that $\forall a\in A,\ I(a)=a$.

- Some identity functions you've seen:
  - $+$ing 0, $\times$ing by 1,
  - $\wedge$ing with **T**, $\vee$ing with **F**,
  - $\cup$ing with $\varnothing$, $\cap$ing with U.

- Remark. The identity function is always both one-to-one and onto (bijective).

Identity Function Illustrations

- The identity function:



Graphs of Functions

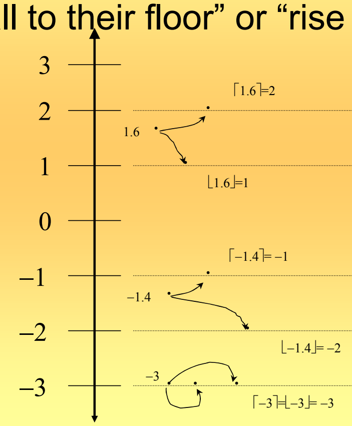
- We can represent a function f: $A \rightarrow B$ as a set of ordered pairs $\{(a,f(a)) \mid a \in A\}$.
- Note that $\forall a$, there is only 1 pair (a,b).
  - Later (ch.8): ***relations* loosen this restriction.**
- For functions over numbers, we can represent an ordered pair (x,y) as a point on a plane.
  - A function is then drawn as a curve (set of points), **with only one *y* for each *x***.

A Couple of Key Functions

- In discrete math, we will frequently use the following two functions over real numbers:
-
- Def. The ***floor*** function $\lfloor \cdot \rfloor : \mathbb{R} \rightarrow \mathbb{Z}$, where $\lfloor x \rfloor$ ("floor of x") **means the largest (most positive) integer $i \leq x$**. *Formally,* $\lfloor x \rfloor :\equiv \max(\{i \in \mathbf{Z} \mid i \leq x\})$.
- Def. The ***ceiling*** function $\lceil \cdot \rceil : \mathbb{R} \rightarrow \mathbb{Z}$, where $\lceil x \rceil$ ("ceiling of x") **means the smallest (most negative) integer $i \geq x$**. *Formally,* $\lceil x \rceil :\equiv \min(\{i \in \mathbf{Z} \mid i \geq x\})$

Visualizing Floor & Ceiling

- Real numbers "fall to their floor" or "rise to their ceiling."
- Note that if $x \notin \mathbf{Z}$, $\lfloor -x \rfloor \neq -\lfloor x \rfloor$ & $\lceil -x \rceil \neq -\lceil x \rceil$
- Note that if $x \in \mathbf{Z}$, $\lfloor x \rfloor = \lceil x \rceil = x$.

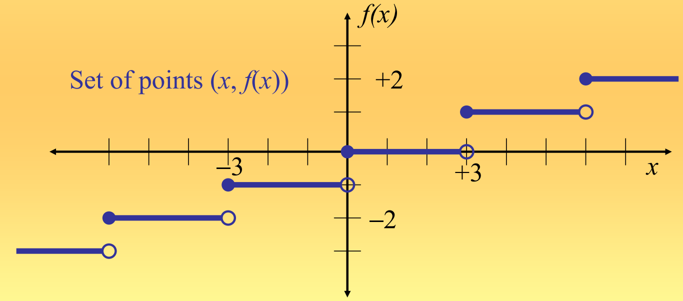


Plots with floor/ceiling

- Note that for $f(x)=\lfloor x \rfloor$, the graph of f includes the point $(a, 0)$ for all values of a such that $a \geq 0$ and $a<1$, but not for the value $a=1$.
- We say that the set of points $(a,0)$ that is in f does not include its *limit* or *boundary* point $(a,1)$.
  - Sets that do not include all of their limit points are generally called *open sets*.
- In a plot, we draw a limit point of a curve using an open dot (circle) ○ if the limit point is not on the curve, and with a closed (solid) ● dot if it is on the curve.

Plots with floor/ceiling: Example

- Plot of graph of function $f(x) = \lfloor x/3 \rfloor$:



Review of §2.3 (Functions)

- Function variables f, g, h, …
- Notations: f: $A \rightarrow B$, $f(a)$, $f(A)$.
- Terms: image, preimage, domain, codomain, range, one-to-one, onto, strictly (in/de)creasing, bijective, inverse, composition.
- Function unary operator f^{-1}, binary operators $+$, $-$, *etc.*, and $\circ$.
- The **R**→**Z** functions $\lfloor x \rfloor$ and $\lceil x \rceil$.

References

- Rosen
  Discrete Mathematics and its Applications, 6e, Mc GrawHill, 2007