VCU, Department of Computer Science

CMSC 302

Sequences and Summations

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Sequences

Rosen 6th ed., §2.4

§2.4: Sequences, Strings, & Summations

- A *sequence* or *series* is just like an ordered *n*-tuple, except:
  - Each element in the series has an associated ***index* number**.
  - A sequence or series may be infinite.
- A *string* is a sequence of *symbols* from some finite *alphabet*.
- A *summation* is a compact notation for the sum of all terms in a (possibly infinite) series.

Sequences

- Def. A *sequence* or *series* $\{a_n\}$ is identified with a ***generating function*** $f: S \to A$ for some subset $S \subseteq \mathbb{N}$ and for some set A.
  - Often we have $S = \mathbb{N}$ or $S = \mathbb{Z}^+ = \mathbb{N} - \{0\}$.
  - Sequences may also be generalized to *indexed sets*, in which the set S does *not* have to be a subset of $\mathbb{N}$.
    - For general indexed sets, S may not even be a set of numbers at all.
- Def. If f is a generating function for a series $\{a_n\}$, then for $n \in S$, the symbol a_n denotes $f(n)$, also called ***term n*** of the sequence.
  - The *index* of a_n is n. (Or, often i is used.)
- A series is sometimes denoted by listing its first and/or last few elements, and using ellipsis (…) notation.
  - *E.g.*, "$\{a_n\} = 0, 1, 4, 9, 16, 25, \ldots$" is taken to mean
  - $\forall n \in \mathbb{N}^+, a_n = (n-1)^2$. it is an **infinite** sequence

Sequence Examples

- Some authors write "the sequence (i.e., series) $a_1, a_2, \ldots$" instead of $\{a_n\}$, to ensure that the set of indices is clear.
  - Be careful: Our book often leaves the indices ambiguous.
- Ex. An example of an **infinite** series:
  - Consider the series $\{a_n\} = a_1, a_2, \ldots$, where $(\forall n \geq 1)\ a_n = f(n) = 1/n$.
  - Then, we have $\{a_n\} = 1, 1/2, 1/3, \ldots$

Example with Repetitions

- Like tuples, **but unlike sets**, a sequence may contain ***repeated*** instances of an element.
- Consider the sequence $\{b_n\} = b_0, b_1, \ldots$ (note that 0 is an index) where $b_n = (-1)^n$.
  - Thus, $\{b_n\} = 1, -1, 1, -1, \ldots$
    - Note repetitions!
  - This $\{b_n\}$ denotes an infinite sequence of 1's and -1's, *not* the 2-element set $\{1, -1\}$.

Recognizing Sequences

- Sometimes, you're given the first few terms of a sequence,
  - and you are asked to find the sequence's generating function,
  - or a procedure to enumerate the sequence.
- Examples: What's the next number?
  - 1,2,3,4,… 5 (the 5th smallest number >0)
  - 1,3,5,7,9,… 11 (the 6th smallest odd number >0)
  - 2,3,5,7,11,... 13 (the 6th smallest prime number)
  - 0,3,8,15, … 24 ($f(n) = n^2 - 1$)

The Trouble with Sequence Recognition

- As you know, these problems are popular on IQ tests, but…
- The problem of finding "the" generating function given just an initial subsequence **is *not a mathematically well defined, i.e., posed, problem.***
  - This is because there are *infinitely* many computable functions that will generate *any* given initial subsequence.
- We implicitly are supposed to find the *simplest* such function (because this one is assumed to be most likely), but,
  - how are we to objectively define the *simplicity* of a function?
- We might define simplicity as the reciprocal of complexity, but…
  - There are *many* different plausible, competing definitions of complexity, and this is an active research area.
- So, these questions really have *no* objective right answer!
  - Still, we will ask you to answer them anyway… (Because others will too.)

Example of Ill-Posedness of a Sequence Recognition

- 0,3,8,15, …is a series
- Solution 1 $24\ (f(n) = n^2 - 1\ )$
- Solution 2 $f(n) = \sum_{i=1}^{4} w_i \sin(i*n), \quad \mathbf{w} = [5.94, \ -37.71, \ -77.37, \ -53.13]$
- Solution 3 $f(n) = \sum_{i=1}^{4} w_i e^{i*n}, \quad \mathbf{w} = [-0.38, 0.16, -0.01, 0.0001]$
- In fact instead of *sin* and *exp* we can use any other function

What are Strings, Really?

- This book says "**finite** sequences of the form $a_1, a_2, \ldots, a_n$ are called *strings*",
  - but ***infinite*** strings are also discussed sometimes.
- Strings are normally restricted to sequences composed of *symbols* drawn from a finite *alphabet*, and are often indexed from 0 or 1.
  - But these are really arbitrary restrictions also.
- Either way, the *length* of a (finite) string is just its number of terms (or of distinct indices).

Strings, more formally

- Def. Let Σ be a finite set of *symbols*, *i.e.* an *alphabet*.

A *string* ***s*** over alphabet Σ is any sequence $\{s_i\}$ of symbols, $s_i \in \Sigma$, normally indexed by $\mathbb{N}$ or $\mathbb{N}-\{0\}$.

- Notation. If *a*, *b*, *c*, … are symbols, the string *s* = *a*, *b*, *c*, … can also be written *abc*…(*i.e.*, without commas).

Strings, more formally

- Def. If *s* is a finite string and *t* is any string, then the *concatenation of s with t*, written just *st*,
- is simply the string consisting of the symbols in *s*, in sequence, followed by the symbols in *t*, in sequence.

More Common String Notations

- Def. The length $|s|$ of a finite string s is its number of *positions* (*i.e.*, its number of index values i).
- Def. If s is a finite string and $n \in \mathbb{N}$, then s^n denotes the concatenation of n copies of s.
  $s = ab$, $s^4 = abababab$
- ε or “” denotes the empty string, the string of length 0. This is fairly common, but the book uses λ instead.
- Def. If Σ is an alphabet and $n \in \mathbb{N}$,
  $\Sigma^n :\equiv \{s \mid s$ is a string over Σ of **length n**$\}$, and
  $\Sigma^* :\equiv \{s \mid s$ is a finite string over $\Sigma\}$.

Example

- Σ is English alphabet and $n = 3 \in \mathbb{N}$

$$\sum^{n} = \{"abc", "abd", "xds", \cdots, "xyz"\}$$

$$\sum^{*} = \{"a", "bc", "abde", "abepr", \cdots, "xyz"\}$$

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Summations

Summation Notation

- Def. Given a series $\{a_n\}$, an integer *lower bound* (or *limit*) $j \geq 0$, and an integer *upper bound* $k \geq j$, then the *summation of $\{a_n\}$ from j to k* is written and defined as follows:

$$\sum_{i=j}^{k} a_i :\equiv a_j + a_{j+1} + \ldots + a_k$$

Here, i is called the *index of summation*.

Generalized Summations

- Notation.
  For an infinite series, we may write:

$$\sum_{i=j}^{\infty} a_i :\equiv a_j + a_{j+1} + \ldots$$

- To sum a function over all members of a set $X=\{x_1, x_2, \ldots\}$:

$$\sum_{x \in X} f(x) :\equiv f(x_1) + f(x_2) + \ldots$$

Simple Summation Example

- $$\sum_{i=2}^{4} \left(i^2+1\right) = (2^2+1) + (3^2+1) + (4^2+1)$$
  $$= (4+1) + (9+1) + (16+1)$$
  $$= 5 + 10 + 17$$
  $$= 32$$

More Summation Examples

- An infinite series with a finite sum:

$$\sum_{i=0}^{\infty} 2^{-i} = 2^0 + 2^{-1} + \ldots = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \ldots = 2$$

- Using a **predicate** to define a set of elements to sum over:

$$\sum_{(x \text{ is prime}) \wedge x<10} x^2 = 2^2 + 3^2 + 5^2 + 7^2 = 4 + 9 + 25 + 49 = 87$$

Note, this is a set {2 3 5 7}

Summation Manipulations

- Some handy identities for summations:

$$\sum_x cf(x) = c\sum_x f(x)$$ (Distributive law)

$$\sum_x \left(f(x)+g(x)\right) = \sum_x f(x) + \sum_x g(x)$$ (An application of commutativity)

$$\sum_{i=j}^{k} f(i) = \sum_{i=j+n}^{k+n} f(i-n)$$ (Index shifting)

More Summation Manipulations

- Other identities that are sometimes useful:

$$\sum_{i=j}^{k} f(i) = \left(\sum_{i=j}^{m} f(i)\right) + \sum_{i=m+1}^{k} f(i) \quad \text{if } j \le m < k$$

(Series splitting)

$$\sum_{i=0}^{k} f(i) = \sum_{i=0}^{k} f(k-i)$$

(Order reversal)

$$\sum_{i=0}^{2k} f(i) = \sum_{i=0}^{k}\big(f(2i) + f(2i+1)\big) - f(2k+1)$$

(Grouping)

Example: Impress Your Friends

- Boast, "I'm so smart; give me any 2-digit number n, and I'll add all the numbers from 1 to n in my head in just a few seconds."
- *i.e.*, Evaluate the summation: $\sum_{i=1}^{n} i$
- There is a simple closed-form formula for the result, discovered by Euler at age 12!
  - And frequently rediscovered by many…

Leonhard Euler (1707-1783)

Euler's Trick, Illustrated

- Consider the sum:
  1+2+…+(n/2)+((n/2)+1)+…+(n-1)+n

  n+1
  ⋮
  n+1
  n+1

- We have n/2 pairs of elements, each pair summing to n+1, for a total of (n/2)(n+1), or **n(n+1) / 2** !!!

Symbolic Derivation of Trick

For case where n is even…

$$\sum_{i=1}^{n} i = \sum_{i=1}^{2k} i = \left(\sum_{i=1}^{k} i\right) + \sum_{i=k+1}^{n} i = \left(\sum_{i=1}^{k} i\right) + \sum_{i=0}^{n-(k+1)} (i+(k+1))$$

$$= \left(\sum_{i=1}^{k} i\right) + \sum_{i=0}^{n-(k+1)} ((n-(k+1))-i)+(k+1))$$

$$= \left(\sum_{i=1}^{k} i\right) + \sum_{i=0}^{n-(k+1)} (n-i) = \left(\sum_{i=1}^{k} i\right) + \sum_{i=1}^{n-k} (n-(i-1))$$

$$= \left(\sum_{i=1}^{k} i\right) + \sum_{i=1}^{n-k} (n+1-i) = \left(\sum_{i=1}^{k} i\right) + \sum_{i=1}^{k} (n+1-i) = \ldots$$

Concluding Euler's Derivation

$$\sum_{i=1}^{n} i = \left(\sum_{i=1}^{k} i\right) + \sum_{i=1}^{k}(n+1-i) = \sum_{i=1}^{k}(i+n+1-i)$$
$$= \sum_{i=1}^{k}(n+1) = k(n+1) = \tfrac{n}{2}(n+1)$$
$$= n(n+1)/2$$

- So, you only have to do 1 easy multiplication in your head, then cut in half.
- Also works for odd *n* (prove this at home).

Geometric Progression

- Def. A *geometric progression* is a series of the form $a, ar, ar^2, ar^3, \ldots, ar^k$, where $a, r \in \mathbb{R}$.
- The sum of such a series is given by:

$$S = \sum_{i=0}^{k} ar^i$$

- We can reduce this to *closed form* via clever manipulation of summations...

Geometric Sum Derivation

- Here we go...

$$S = \sum_{i=0}^{n} ar^i$$
$$rS = r\sum_{i=0}^{n} ar^i = \sum_{i=0}^{n} rar^i = \sum_{i=0}^{n} arr^i = \sum_{i=0}^{n} ar^1 r^i$$
$$= \sum_{i=0}^{n} ar^{1+i} = \sum_{i=1}^{n+1} ar^{1+(i-1)} = \sum_{i=1}^{n+1} ar^i$$
$$= \left(\sum_{i=1}^{n} ar^i\right) + \sum_{i=n+1}^{n+1} ar^i = \left(\sum_{i=1}^{n} ar^i\right) + ar^{n+1} = \ldots$$

Geometric Sum Derivation ...

- $$rS = \left(\sum_{i=1}^{n} ar^i\right) + ar^{n+1} = (ar^0 - ar^0) + \left(\sum_{i=1}^{n} ar^i\right) + ar^{n+1}$$
$$= ar^0 + \left(\sum_{i=1}^{n} ar^i\right) + ar^{n+1} - ar^0$$
$$= \left(\sum_{i=0}^{0} ar^i\right) + \left(\sum_{i=1}^{n} ar^i\right) + ar^{n+1} - a$$
$$= \left(\sum_{i=0}^{n} ar^i\right) + a(r^{n+1} - 1) = S + a(r^{n+1} - 1)$$

Geometric Sum Derivation ...

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$$rS = S + a(r^{n+1} - 1)$$
$$rS - S = a(r^{n+1} - 1)$$
$$S(r-1) = a(r^{n+1} - 1)$$
$$\boxed{S = a\left(\frac{r^{n+1}-1}{r-1}\right) \quad \text{when } r \neq 1}$$
$$\text{When } r = 1,\ S = \sum_{i=0}^{n} ar^i = \sum_{i=0}^{n} a1^i = \sum_{i=0}^{n} a \cdot 1 = (n+1)a$$

Nested Summations

- These have the meaning you'd expect.

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}\left(\sum_{j=1}^{3} ij\right) = \sum_{i=1}^{4} i\left(\sum_{j=1}^{3} j\right) = \sum_{i=1}^{4} i(1+2+3)$$
$$= \sum_{i=1}^{4} 6i = 6\sum_{i=1}^{4} i = 6(1+2+3+4)$$
$$= 6 \cdot 10 = 60$$

- Note **issues of free** vs. **bound variables**, just like in quantified expressions, integrals, etc.

Some Shortcut Expressions

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$$\sum_{k=0}^{n} ar^k = a(r^{n+1}-1)/(r-1), r \neq 1$$ Geometric series

$$\sum_{k=1}^{n} k = n(n+1)/2$$ Euler's trick

$$\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$$ Quadratic series

$$\sum_{k=1}^{n} k^3 = n^2(n+1)^2/4$$ Cubic series

Using the Shortcuts

- Example: Evaluate $\sum_{k=50}^{100} k^2$.
  - Use series splitting.
  - Solve for desired summation.
  - Apply quadratic series rule.
  - Evaluate.

$$\sum_{k=1}^{100} k^2 = \left(\sum_{k=1}^{49} k^2\right) + \sum_{k=50}^{100} k^2$$
$$\sum_{k=50}^{100} k^2 = \left(\sum_{k=1}^{100} k^2\right) - \sum_{k=1}^{49} k^2$$
$$= \frac{100 \cdot 101 \cdot 201}{6} - \frac{49 \cdot 50 \cdot 99}{6}$$
$$= 338{,}350 - 40{,}425$$
$$= 297{,}925.$$

Summations: Conclusion

- You need to know:
  - How to read, write & evaluate summation expressions like:

$$\sum_{i=j}^{k} a_i \qquad \sum_{i=j}^{\infty} a_i \qquad \sum_{x \in X} f(x) \qquad \sum_{P(x)} f(x)$$

  - Summation manipulation laws we covered.
  - Shortcut closed-form formulas, & how to use them.

References

- Rosen
  Discrete Mathematics and its Applications, 6th ed., Mc GrawHill, 2007